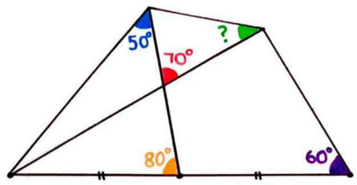
October Math Gems

Problem of the week 2

§1 problems

Problem 1.1. Find the unknown value.



Solution. we will use the sine rule

$$\frac{KB}{\sin \angle A} = \frac{AB}{\sin \angle AKB} = \frac{KA}{\sin \angle KBA}$$

$$KA = \frac{\sin 80 \times AB}{\sin 50}$$

As

$$AB = \frac{1}{2} \times AC$$

$$KA = \frac{\sin 80 \times AC}{2 \times \sin 50}$$

Now we will use the cosine rule in $\triangle AKD$

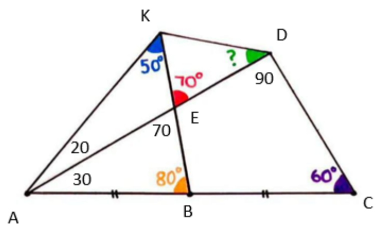
$$KD = \sqrt{(AK)^2 + (AD)^2 - 2 \times AK \times AD \times \cos 20}$$

$$\frac{\sin x}{AK} = \frac{\sin KAD}{KD} \implies \sin x = \frac{\sin 20 \times AK}{KD}$$

So,

$$x = 40$$

□



Problem 1.2. Find the maximum obtainable value by adding a single set of parentheses into the expression

$$1 + 2 \times 3 + 4 \times 5 + 6$$

Solution. The maximum value is obtained when the parentheses are placed as follows

$$1 + 2 \times (3 + 4) \times 5 + 6$$

This yields a value of 77 □

Problem 1.3. Evaluate the following expression

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + ... + \frac{1}{\sqrt{2016}+\sqrt{2017}}$$

Solution. First, we multiply each fraction by its conjugate, and replace all the denominators by -1. This simplifies to the following expression

$$\frac{1 - \sqrt{2} + \sqrt{2} - \sqrt{3} + \sqrt{3} - \sqrt{4} + \sqrt{4} - ... + \sqrt{2016} - \sqrt{2017}}{-1}$$

This expression simplifies to $\frac{1-\sqrt{2017}}{-1}$, which is equal to $\sqrt{2017} - 1$ □

Problem 1.4. Evaluate

$$\frac{e}{\sqrt{e}} \times \frac{\sqrt[3]{e}}{\sqrt[4]{e}} \times \frac{\sqrt[5]{e}}{\sqrt[6]{e}} \times ...$$

Where e is the euler number

Solution. First, we will use the identities $(x^{\frac{m}{n}}) = \sqrt[n]{x^m}$, $x^m \times x^n = x^{m+n}$, and $\frac{x^m}{x^n} = x^{m-n}$ to simplify the given expression. Therefore, we can rewrite it as

$$e^{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...}$$

The exponent is equal to ln(2) (The exponent is called an alternating harmonic series where x is equal to 1). Therefore, our final answer is 2 □

Problem 1.5. What is the domain of the following function?

$$\frac{x+2}{x^2+x-2}$$

Solution. First, we factorize the denominator to be $(x+2)(x-1)$. Therefore, $x \neq -2, x \neq 1$ Therefore, the domain is $x \in R$ except $x = -2, 0$ □

Problem 1.6. If the length of a rectangle is decreased by 30 percent, and the width is increased by 15 percent. The area of the new rectangle will be smaller than the original rectangle by what percent?

Solution. First, let the area of the original rectangle be LW. Therefore, the area of the new rectangle is

$$0.7 \times L \times 1.15 \times W$$

Therefore, the area of the new triangle is $0.805LW$. This means that it is 0.195 smaller, which is equal to 19.5 percent smaller. □

Problem 1.7. The side lengths of a scalene triangle are the roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the square of the area of the triangle.

Solution. Let the polynomial be

$$p(x) = x^3 - 20x^2 + 131x - 281.3$$

and let the three side lengths of the triangle be α, β, and γ. Then

$$p(x) = (x - \alpha)(x - \beta)(x - \gamma)$$

Heron's Formula gives the square of the area of the triangle as

$$s(s - \alpha)(s - \beta)(s - \gamma)$$

Where s is the semi-perimeter of the triangle $\frac{\alpha+\beta+\gamma}{2}$. Vieta's Formulas imply that

$$\alpha + \beta + \gamma = 20$$

So $s = 10$. It follows that the requested squared area is

$$10 \cdot p(10) = 287$$

.

The polynomial does have three positive real roots. Note that $p(5) = -1.3 < 0, p(6) = 0.7 > 0, p(7) = -1.3 < 0$, and $p(10) = 28.7 > 0$, implying that the polynomial has roots between 5 and 6, between 6 and 7, and between 7 and 10. Because $5 + 6 > 10$, it follows that α, β, and γ satisfy the triangle inequality, and there is indeed a triangle with these side lengths. The side lengths are approximately 5.2771, 6.4206, 8.3023. □

Problem 1.8.

$$\sqrt{x} + \sqrt{x - \sqrt{1 - x}} = 1$$

Solve for x.

Solution.

$$\sqrt{x} + \sqrt{x - \sqrt{1 - x}} = 1$$

$$\sqrt{x - \sqrt{1 - x}} = 1 - \sqrt{x}$$

$$x - \sqrt{1 - x} = 1 - 2\sqrt{x} + x$$

$$-\sqrt{1 - x} = 1 - 2\sqrt{x}$$

$$1 - x = 1 - 4\sqrt{x} + 4x$$

$$5x - 4\sqrt{x} = 0$$

Suppose that $x = y^2$. So,

$$5y^2 - 4y = 0 \implies y = 0, \quad y = \frac{4}{5}$$

So,

$$x = 0, \quad x = \frac{16}{25}$$

Now, verify the solutions, we find that $x = 0$ give us $\sqrt{-1}$ imaginary number so we will choose □

Problem 1.9. If $a^2 + 1 = 9a$, then compute the value of the following expression

$$a^2 + \frac{1}{a^2}$$

Solution.

$$a^2 + 1 = 9a \implies \frac{a^2+1}{a} = 9$$

$$a + \frac{1}{a} = 9$$

So,

$$(a + \frac{1}{a})^2 = a^2 + 2 + \frac{1}{a^2} = 81$$

$$a^2 + \frac{1}{a^2} = 81 - 2 = 79$$

□

Problem 1.10. If $4b^2 + \frac{1}{b^2} = 2$, then compute the value of the following expression

$$8b^3 + \frac{1}{b^3}$$

Solution. We will use the identity

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Let

$$x = 2b, \qquad y = \frac{1}{b}$$

So,

$$((2b)^3 + (\frac{1}{b})^3) = (8b^3 + \frac{1}{b^3}) = (2b + \frac{1}{b})((2b)^2 - \frac{1}{b} \times 2b + (\frac{1}{b})^2) = (2b + \frac{1}{b})(4b^2 + \frac{1}{b^2} - 2)$$

Replace $4b^2 + \frac{1}{b^2} = 2$ from the givens, So we get

$$8b^3 + \frac{1}{b^3} = (2b + \frac{1}{b}) \times (-2 + 2) = 0$$

□

Problem 1.11. Solve for x

$$x^2 + \frac{9x^2}{(x+3)^2} = 27$$

Solution. Multiply by $(x+3)^2$,

$$x^2(x+3)^2 + 9x^2 = 27(x+3)^2$$

$$x^4 + 6x^3 + 18x^2 = 27x^2 + 162x + 243$$

$$x^4 + 6x^3 - 9x^2 - 162x - 243 = 0$$

By using Newton-Raphson and applying long division we get

$$x = -\frac{3(\sqrt{5}-1)}{2}, \qquad x = \frac{3(\sqrt{5}+1)}{2}$$

□

Problem 1.12. If $x^2 + y^2 - 2x + 6y + 10 = 0$, then compute the value of the following expression
$$(x^2 + y^2)$$
is?

Solution.
$$x^2 + y^2 - 2x + 6y + 1 + 9 = 0$$
$$(x^2 - 2x + 1) + (y^2 + 6y + 9)^2 = 0$$
$$(x - 1)^2 + (y + 3)^2 = 0$$
So we can say that
$$\begin{cases} (x - 1) = 0 \implies x = 1 \\ (y + 3) = 0 \implies y = -3 \end{cases}$$
So
$$x^2 + y^2 = (1)^2 + (-3)^2 = 1 + 9 = 10$$
□

Problem 1.13. Compute
$$\frac{5 + \sqrt{6}}{\sqrt{2} + \sqrt{3}} + \frac{7 + \sqrt{12}}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{63 + \sqrt{992}}{\sqrt{31} + \sqrt{32}}$$

Problem 1.14. If $3a + 4b = 5$, then what is the minimum value of $a^2 + b^2$?

Solution. First,
$$b = \frac{5 - 3a}{4}$$
Therefore, we can write $a^2 + b^2$ as
$$a^2 + \frac{1}{16}(5 - 3a)^2$$
Getting a^2 into the brackets and completing the squares, we get the following equation
$$\frac{25}{16}(a - \frac{3}{5})^2 + 1$$
So, the minimum value is achieved when $a = \frac{3}{5}$, and the whole equation is equal to 1. Therefore, the minimum value for $a^2 + b^2$ is 1. □

Problem 1.15. Evaluate
$$\frac{1}{1 + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + ... + \frac{1}{\sqrt{167} + \sqrt{169}}$$

Solution. First, we can multiply each fraction by its conjugate, and take $\frac{-1}{2}$ as a common factor. Then, we are left with
$$\frac{-1}{2}((1 - \sqrt{3}) + (\sqrt{3} - \sqrt{5}) + (\sqrt{5} - \sqrt{7}) + ... + (\sqrt{167} - \sqrt{169}))$$
After simplifying, we are left with
$$\frac{-1}{2}(1 - \sqrt{169})$$
This is equal to 6, which is our final answer. □

Problem 1.16. Let a, b and c are distinct nonzero real numbers where,

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$$

Then, find the value of $(abc)^{2022}$

answer. The only solution for this problem is $a = b = c = 1$ which yields the answer to 1 □

Problem 1.17. Compute

$$\log_{a^a} b^b \times \log_{b^c} c^a \times \log_{c^b} a^c$$

answer. The expression can be written as follow

$$\frac{b \log b}{a \log a} \times \frac{a \log c}{c \log b} \times \frac{c \log a}{b \log c} =$$

$$\frac{b \log b}{b \log b} \times \frac{a \log a}{a \log a} \times \frac{c \log c}{c \log c} = 1$$

□

Problem 1.18. If $x, y, z > 0$ such that $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ Compute

$$x^x y^y z^z$$

Proof. Let

$$\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y} = k$$

Then,

$$\log x = ky - kz$$

$$\log y = kz - kx$$

$$\log z = kx - ky$$

Multiply the first equation by x for both sides, the second equation by y for both sides, and the third equation by z for both sides.

$$\log x^x = kyx - kzx$$

$$\log y^y = kzy - kxy$$

$$\log z^z = kxz - kyz$$

Add the three equations together

$$\log x^x + \log y^y + \log z^z = kyx - kzx + kzy - kxy + kxz - kyz$$

$$\log x^x y^y z^z = 0 \qquad \rightarrow \qquad x^x y^y z^z = 1$$

□

Problem 1.19. Solve the following equation

$$x^{2\ln(x)} = 10x^2$$

Solution. Taking logarithm of the base 10 for both sides yields:

$$2\log^2(x) = \log(10x^2)$$

$$2\log^2(x) = \log(10) + 2\ln(x)$$

$$2\log^2(x) - 1 - 2\log(x) = 0$$

Notice that this equation is in the quadratic form, so we simply can use the quadratic formula.

$$\log x = 1 \pm \sqrt{3}$$

$$x = 10^{1\pm\sqrt{3}}$$

□

Problem 1.20. Compute

$$\log_5 \sqrt{5\sqrt{5\sqrt{5\sqrt{5\sqrt{5....\infty}}}}}$$

Solution. First, to make the problem more easier, we'll work on computing the interior radical. Notice this radical can be written as

$$y = \sqrt{5y}$$

Now, we can square both sides

$$y^2 = 5y$$

$$y^2 - 5y$$

Notice that this equation gives us two solutions, the first that $y = 0$ which is refused because 0 is not included in the domain of the logarithmic functions, the second solution is that $y = 5$, so

$$\log_5 5 = 1$$

□